MODULE 5: RATIONAL NUMBERS, PART 2

COMMON FRACTIONS, CONTINUED

In Module 4 we talked about such ratios as:

$\frac{1}{2}$ of 4 *(4 ÷ 2)*

$\frac{1}{3}$ of 15 *(15 ÷ 3)*

$\frac{1}{4}$ of 24 *(24 ÷ 4)*

$\frac{1}{5}$ of 35 *(35 ÷ 5)*

In each of these division problems we made sure that the dividend was a multiple of the divisor.

For example, we omitted such problems as:

$\frac{1}{2}$ of 9 *(9 ÷ 2)*

$\frac{1}{3}$ of 11 *(11 ÷ 3)*

$\frac{1}{4}$ of 33 *(33 ÷ 4)*

$\frac{1}{5}$ of 42 *(42 ÷ 5)*

Nevertheless, there are times when it is important to divide one number by another, even if the quotient isn't a whole number.

To say that the quotient isn't a whole number means that when we do the divisic there is a non-zero remainc

Example 1

The cost of a $9 gift is shared equally by 2 people. How much money did each person pay?

Answer: $4.50

The numerical answer will modify the

phrase "dollars *per* person" or

"dollars ÷ person(s)"

Recall that at the end of Module 3 we mentioned that "per" between two nouns ma be read arithmetically as "divided by". For example "miles per hour" means "miles divided by hours"

So we take the number of dollars (9) and divide this by the number of people (2).

In this case we can avoid remainders by using the fact that there are 100 cents per dollar. That is, instead of dividing 9 (dollars) by 2 we can divide 900 (cents) by 2 to get:

900 cents ÷ 2 people =

450 cents per person.

We divide the numbers as we would have in Module 3, and we divide "cents" by "people" by writing"cents per person" (which is the accepted usage for "cents per people).

We then write 450 cents in the more usual form of \$4.50 and obtain the answer to Example 1.

In the language of fractional parts, 1/2 of \$9 is \$4.50

However, we don't always have the same success with other problems that have the same format as Example 1.

<u>Example 2</u>

The cost of a \$5 gift is to be shared by 3 people. Is it possible for all 3 to pay exactly the same amount?

Answer: No

\$5 is the same as 500 cents. If the 3 people each pay the same amount, each would pay $\frac{1}{3}$ of 500 cents. Therefore the number of cents per person would be

$\frac{1}{3}$ of 498 cents is 166 cents.

$\frac{1}{3}$ of 501 cents is 167 cents. So $\frac{1}{3}$ of 500 cents is more than 166 cents, yet less than 167 cents.

500 ÷ 3 or:

$$\begin{array}{r} 166\ \text{R2} \\ 3\overline{)500} \\ -3 \\ \hline 20 \\ -18 \\ \hline 20 \\ -18 \\ \hline 2 \end{array}$$

Since the remainder is not 0, the quotient is not a whole number.

What is important here is that in our monetary system, there is no coin that has less value than a cent. So there is nothing more we can do.

When we had to divide \$5 by 3, the quotient wasn't a whole number either, but we had recourse to writing \$5 as 500 cents. There is no way to similarly "break down" 500 cents.

In real-life we would not be too upset by the answer to Example 2. Two of the people might have agreed to each pay \$1.67 and the third person could have paid \$1.66. Or maybe one person paid \$1.75, another paid \$1.70, and the third \$1.55. *But the important point is that it is impossible for 3 people to share the cost of a \$5 gift in a way that the shares are exactly the same.*

This is a real-life version of "rounding off". We often divide costs in this way.

\$1.67	*\$1.75*
\$1.67	*\$1.70*
\$1.66	*\$1.55*
\$5.00	*\$5.00*

Other times, fractional units allow us to divide 5 by 3 without a remainder.

Example 3

3 People decide to share a 5 hour shift so that all 3 work exactly the same length of time. How long does each person work?

Answer: 100 minutes (or 1 hour and 40 minutes)

We want to divide 5 (hours) by 3 (people). But in this case there are 60 minutes per hour. So in 5 hours, there are 60 minutes, 5 times; or 300 minutes. Hence:

5 hours ÷ 3 people =

300 minutes ÷ 3 people =

100 minutes per person

In Example 2 we replaced \$1 by 100¢. The prolem was that 100 isn't divisible by 3 either. In this example we replace 1 hour by 60 minutes, and 60 is divisible by 3.

300 ÷ 3 = 100
minutes ÷ people = minutes per person

100 minutes
-60 minutes (= 1 hour)
40 minutes

Hence 100 minutes = 1 hour and (plus) 40 minutes

SPECIAL NOTE:

It is unrealistic to ask each person to work exactly the same length of time. So if 3 people work a total of 300 minutes, we prefer to say that the average *time each person worked was 100 minutes. That is, if we divide 300 minutes by 3 people, we say that the* average rate *is 100 minutes per person.*

Remember that a rate is indicated whenever two nouns are separated by the word "per".

Example 4

4 apples cost \$1. What is the average cost of each apple?

Answer: \$0.25 (25¢)

The label "cost per apple" translates into "cost ÷ apple(s)" and this reminds us to divide the cost (\$1 or 100 cents) by 4 (apples). We get:

\$1 ÷ 4 apples =

100 cents ÷ 4 apples =

25 cents per apple.

We have to use the phrase "average cost per apple" because we were not told that the apples were equally priced. *If the apples are equally priced then the cost of each apple would be 25¢. That is, when the objects are equally priced, the average price is the same as the price of each object.*

For example, the apples may have been sold by the pound, in which case bigger apples would cost more than smaller ones. For example:

\$0.29
\$0.23
\$0.26
\$0.22
\$1.00

So any 4 apples that cost a total of \$1 would cost an average *of 25¢ per apple.*

Returning now to our main theme, the point is that using tally marks to interpret numbers limits our ability to deal with fractional parts adequately. For example, it is easy to use tally marks to

illustrate why half of 6 is 3. We simply take a group of 6 tally marks and divide them into 2 groups of 3.

"half of 6 is 3" can be read as "1/2 of 6 is 3" or "6÷ 2 = 3"

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But there is no way to interpret 5 ÷ 2 using tally marks. You might be tempted to break one of the tally marks in half, but if you do this, you get two tally marks. That is

For example, if you break a piece of chalk in half you get 2 smaller pieces of chalk.

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is 5 tally marks, but if we break the last one into 2 equal parts, we get:

When you count, say, the number of people in a room, you do not count one person as a fraction because he is smaller than another person. For example, when we say that there are 25 people in a room we ignore the relative size of each person

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which is *six* tally marks. In other words:

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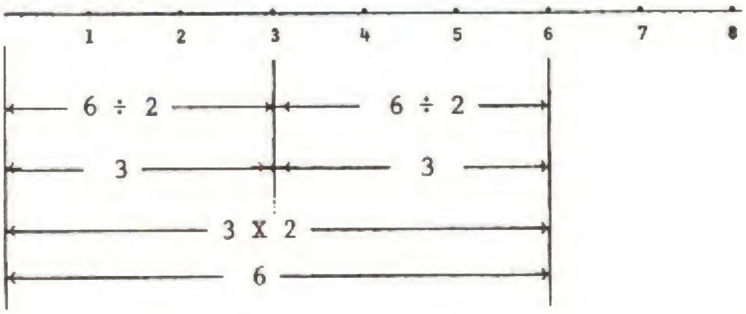
is still an illustration of dividing 6 by 2.

Because tally marks became inadequate for interpreting certain important division problems, the ancient Greeks came up with an interesting new interpretation of whole numbers. *They viewed whole numbers as <u>lengths</u>. Using lengths gave them an easier way to picture <u>rational numbers</u>. Lengths allow us the advantages of tally marks but eliminate many of the disadvantages.*

This is pretty much what we do today when we use a ruler

For example, we can think of 6 ÷ 2 as being the length of each piece if a 6 inch length is divided into 2 *equal* parts.

" inches" isn't important. What is important is that we have 6 <u>units of length</u>, be they inches, feet, meters, or furlongs.

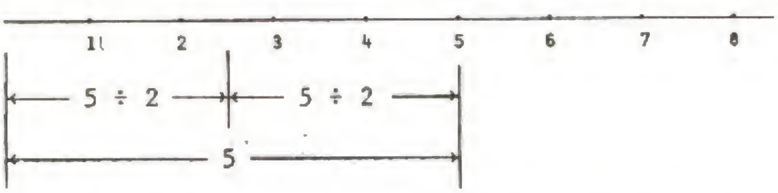


In this interpretation 6 ÷ 2 and 3 are different names for the same length. (In this diagram each unit of length is 5 spaces on my typewriter. So we could say that 5 spaces represented 1 inch.)

While this diagram may be bit more complicated than the tally mark interpretation, the fact is that the diagram is used exactly the same way when we want to divide 5 by 2. That is, we can use lengths in exactly the same way to get an exact length as an answer to 5 ÷ 2. This time we start with a 5 unit length and divide it into 2 equal lengths. That is:

Remember that we only want a more complicated way when the complicated way brings us results we can't get by the "easier" way.

In fact, we can divide any length into 2 equal parts. We often do this with string or thread. We pick up the string and place its ends together.



See the idea? While we can't have more than 2 tally marks but fewer than 3, there are many lengths that are longer than 2 inches but shorter than 3 inches.

Using this interpretation of length, there was a slightly different origin to the notion of common fractions.

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Alternative Definition of Common Fraction

If m and n are any whole numbers, and m is not 0; we write $\frac{m}{n}$ to stand for m ÷ n.

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In this interpretation the numerator is the dividend and the denominator is the divisor.

Let's see how the alternative definition works.

Example 5

What rational number is named by $\frac{6}{2}$?

Answer: 6 ÷ 2 or 3

$\frac{6}{2}$ stands for 6 ÷ 2, and 6 ÷ 2 = 3.

$$\frac{6}{2} \rightarrow 6 \div 2$$

Note

Beware of the order. We always divide the numerator by the denominator. That is:

$$\frac{\textit{numerator}}{\textit{denominator}} \textit{ means numerator} \div \textit{denominator}$$

In terms of money, it makes a difference whether 5 peop share a \$2 gift or whether 2 people share a \$5 gift.

Another way to remember this is to leave the denominator in place and "swing down" the numerator to create the division problem:

$$\frac{\textit{numerator}}{\textit{denominator}\,)\,\overline{\textit{numerator}}}$$

So we see that the denomina tor is the divisor and the numerator is the dividend.

Example 5 shows us how the new definition of common fraction is related to how we used common fractions in Module 4. Namely, $\frac{6}{2}$ means the same thing as $\frac{1}{2}$ of 6.

That is, to divide 6 by 2 is the same as taking half of 6.

In a similar way we can think of $\frac{3}{5}$ as meaning the same thing as $\frac{1}{5}$ of 3. *It is sometimes more helpful if we think of $\frac{3}{5}$ as meaning $\frac{3}{5}$ of 1.* This idea may become more clear after we look at the next example.

We've talked about numbers being adjectives. So when we say, for example, 3/5; we must indicate 3/5 of wha Thus we talk about 3/5 of 60 or 3/5 of 76. When we simp say "3/5" we shall mean "3/5 of 1"

Example 6

(a) The cost of a \$3 item is shared equally by 5 people. How much does each person pay?

(b) How much is $\frac{3}{5}$ of \$1?

Answers: (a) 60¢ (b) 60¢

(a) We have:

\$3 ÷ 5 people =

300¢ ÷ 5 people =

60¢ per person

(b) $\frac{3}{5}$ of \$1 =

$\frac{3}{5}$ of 100¢

So in terms of what we did in Module 4, we divide 100¢ by 5 to get 20¢; and we then multiply 20¢ by 3 to get 60¢

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Key Point

If m and n are whole numbers, then:

$$\frac{m}{n},\ \frac{1}{n} \text{ of } m, \text{ and } \frac{m}{n} \text{ of } 1$$

all mean the same thing.

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For example, $\frac{4}{7}$, $\frac{1}{7}$ of 4 and $\frac{4}{7}$ of 1 all mean the same.

The word "of" as we use it in the phrase "$\frac{3}{5}$ of 100" has a strong connection with multiplication when we deal with fractional parts. For example, we often want to take a fractional part of a fractional part. Let's see what it means to compute, say, $\frac{3}{5}$ of $\frac{4}{7}$.

This often happens in sales. A store might take 20% off the price of an object. Later to make the item more attractive the store might take off 20% of the remaining price. (Percent will be discussed in the next module. We're using it here only as an illustration).

The denominators hint to us that we shall ultimately be dividing one amount by 7 and another amount by 5. So in order to work with whole numbers, we should start with a common multiple of 5 and 7.

Therefore let's suppose we had 35 items.

Then:

$\frac{4}{7}$ of 35 is 20

We divided 35 by 7 and then multiplied by 4.

Next we take $\frac{3}{5}$ of $\frac{4}{7}$ of 35. We get:

$\frac{3}{5}$ of $\frac{4}{7}$ of 35 =

$\frac{3}{5}$ of ($\frac{4}{7}$ of 35) =

The grouping symbols indicate that we first take 4/7 of 35 and then take 3/5 of that answer.

$\frac{3}{5}$ of 20 =

12

We simply replaced $\frac{4}{7}$ of 35 by 20. Now we divide 20 by 5 and multiply the answer by 3.

So what we showed is that to take $\frac{3}{5}$ of $\frac{4}{7}$ (of 35) means the same thing as taking 12 (of the 35).

There was no reason to start with 35 simply because it was convenient. We could have used any multiple of 35 and got a similar result.

<u>Example 7</u>

How much is $\frac{3}{5}$ of $\frac{4}{7}$ of 105?

Answer: 36

We read the problem as:

$\frac{3}{5}$ of ($\frac{4}{7}$ of 105) =

We divide 105 by 7 to get 15 and we multiply 15 by 4.

$\frac{3}{5}$ of 60 =

We divide 60 by 5 to get 12 and then we multiply 12 by 3.

36

The point is that 12 out of each 35 is the same ratio as 36 out of each 105. In terms of common fractions:

$$\frac{12}{35} = \frac{12 \times 3}{35 \times 3} = \frac{36}{105}$$

This is why we like "lowest terms". It is easier to visualize 12 out of each 35 than 36 out of each 105.

At any rate, let's compare the problem with the answer (in lowest terms) and see if we can recognize a pattern.

$$\frac{3}{5} \text{ of } \frac{4}{7} = \frac{12}{35}$$

It appears that we multiplied the two numerators to get the numerator of the answer (3 X 4 = 12); and that we multiplied the two denominators to get the denominator of the answer (5 X 7 = 35). Because of this strong hint of multiplication, we replace the

word "of" by the "X" and arrive at the following definition of multiplication:

$$\frac{3}{5} \times \frac{4}{7} = \frac{3 \times 4}{5 \times 7}$$

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Multiplying Two Common Fractions:

If $\frac{a}{b}$ and $\frac{c}{d}$ are common fractions, we define their product by:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

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Example 7*(a)*

Write the product of $\frac{2}{3}$ and $\frac{5}{7}$ as a common fraction.

Answer: $\frac{10}{21}$

By our rule of multiplying the two numerators and multiplying the two denominators we get:

$$\frac{2}{3} \times \frac{5}{7} = \frac{2 \times 5}{3 \times 7} = \frac{10}{21}$$

The most important thing to remember is that we use this rule for multiplication *because it agrees with real-world answer*--*NOT BECAUSE IT'S "EASY"*.

For example, in addition we have to get a common denominator. We don't add numerators and denominators even though that would be "easier".

To make sure you understand this, let's do a real-life problem that uses the answer to Example 7.

Example 8

Your friend buys a coat for $\frac{5}{7}$ of the regular price. He sells it to you for $\frac{2}{3}$ of the price *he paid*. What portion of the regular price did you pay?

Answer: $\frac{10}{21}$
(That is, for each $21 of the regular price you pay $10)

Since the denominators of the fractions are 3 and 7, it is easiest to think in terms of 3 X 7 or 21.

So suppose the coat cost \$21. Your friend pays $\frac{5}{7}$ of \$21. Since 21 divided by 7 is 3 and 3 X 5 is 15, your friend pays \$15 (of the \$21)

Be careful here. We are not saying the coat cost \$2. What we're doing is to figure the cost based on <u>each</u> \$21 of the regular price.

You are to pay him $\frac{2}{3}$ of his cost. His cost is \$15, not \$21. Hence, for each \$21 of the regular cost of the coat, you pay

$$\frac{2}{3} \text{ of } \$15.$$

\$15 divided by 3 is \$5 and \$5 X 2 is \$10. Hence you pay \$10 for each \$21 of the regular price. In the language of common fractions, you are paying $\frac{10}{21}$ of the regular price.

In terms of a chart:

Regular Price	*Your Price*
\$21	*\$10*
\$42	*\$20*
\$63	*\$30*
\$84	*\$40*

That is:
$\frac{10}{21} = \frac{20}{42} = \frac{30}{63} = \frac{40}{84}$ *and so on*

(The numerators are multipl of 10 and the denominators are multiples of 21)

The key point is that in Example 8 we wanted $\frac{2}{3}$ of $\frac{5}{7}$ and in Example 7 we solved this problem by showing that $\frac{2}{3} \times \frac{5}{7} = \frac{2 \times 5}{3 \times 7} = \frac{10}{21}$

<u>Special Note:</u>

We can find $\frac{2}{3}$ of $\frac{5}{7}$ without using money. Namely rewrite the problem as:

$$\frac{2}{3} \text{ of 5 sevenths}$$

This emphasizes that we want 2/3 of 5.

The problem is that 5 is not a multiple of 3. To overcome this problem we multiply numerator and denominator of $\frac{5}{7}$ by 3 to get $\frac{15}{21}$. Hence:

$\frac{2}{3}$ *of 5 sevenths =*

$\frac{2}{3}$ *of 15 twenty-firsts =*

10 twenty-firsts =

$\frac{10}{21}$

Get the idea? Since 15 is multiple of 3, we can get the answer as a whole numbe

The rule for multiplying common fractions allows us to use cancellation to simplify certain problems.

Example 9

Write $\frac{4}{5} \times \frac{15}{28}$ as a common fraction in lowest terms.

Answer: $\frac{3}{7}$

We could use the rule for multiplication directly to obtain:

$$\frac{4}{5} \times \frac{15}{28} = \frac{60}{140}$$

To reduce to lowest terms we first notice that both 60 and 140 are multiples of 10. Hence we may divide numerator and denominator by 10 to reduce $\frac{60}{140}$ to $\frac{6}{14}$. Then since both 6 and 14 are divisible by 2, we can reduce $\frac{6}{14}$ to $\frac{3}{7}$

In effect we used the fact that:

$$\frac{3}{7} = \frac{3 \times 20}{7 \times 20} = \frac{60}{140}$$

But we could have used prime factorization to see the common factors more directly.

Namely:

$$\frac{4}{5} \times \frac{15}{28} =$$

$$\frac{4 \times 15}{5 \times 28} =$$

$$\frac{2 \times 2 \times 3 \times 5}{5 \times 2 \times 2 \times 7}$$

We used the facts that $4 = 2 \times 2$; $15 = 3 \times 5$; *and* $28 = 2 \times 2 \times 7$

We may cancel the first 2 in the numerator with the first 2 in the denominator to obtain:

$$\frac{\cancel{2} \times 2 \times 3 \times 5}{5 \times \cancel{2} \times 2 \times 7}$$

By the commutative property of multiplication we do not have to "line up" the common factors directly over one another in order to cancel.

Then we may cancel the second 2 in the numerator with the second 2 in the denominator to obtain:

$$\frac{\cancel{2} \times \cancel{2} \times 3 \times 5}{5 \times \cancel{2} \times \cancel{2} \times 7}$$

Finally we may cancel the 5 from the numerator with the 5 from the denominator to obtain: $\frac{\cancel{2} \times \cancel{2} \times 3 \times \cancel{5}}{\cancel{5} \times \cancel{2} \times \cancel{2} \times 7}$

and we obtain as our reduced answer, $\frac{3}{7}$.

Since 3 and 7 are both prime numbers, we cannot cancel further.

Since we multiply numerators and multiply denominators when we multiply fractions, we can cancel any common factor that exists between <u>any</u> numerator and denominator. For example, we could have cancelled the 4 in 4 with a 4 in 28 to get:

$$\frac{\overset{1}{\cancel{4}}}{5} \times \frac{15}{\underset{7}{\cancel{28}}}$$

In the form $\frac{4 \times 15}{5 \times 28}$ it is easier to see that 4 is a common factor of both the numerator and the denominator. The same goes for 5.

Then we could have cancelled a 5 in the first denominator with a 5 in the second numerator to obtain:

$$\frac{\overset{1}{\cancel{4}}}{\underset{1}{\cancel{5}}} \times \frac{\overset{3}{\cancel{15}}}{\underset{7}{\cancel{28}}}$$

We then would multiply 1 by 3 to get 3, and 1 by 7 to get 7; and our final answer would be $\frac{3}{7}$.

The arithmetic properties that applied to whole numbers also apply to rational numbers.

<u>Example 10</u>

How much is $\frac{2}{3} \times 1$?

Answer: $\frac{2}{3}$

$\frac{2}{3} \times 1$ means $\frac{2}{3}$ of 1; and we've already mentioned that $\frac{2}{3}$ of 1 means the same thing as $\frac{2}{3}$.

Before our property was $n \times 1 = n$ if n is any whole number; now the property holds for any rational number (common fraction)

Example 11

Which is the greater ratio,
$\frac{2}{3} \times \frac{5}{7}$ or $\frac{5}{7} \times \frac{2}{3}$?

Answer: They're equal

$$\frac{2}{3} \times \frac{5}{7} = \frac{2 \times 5}{3 \times 7} = \frac{10}{21}$$

$$\frac{5}{7} \times \frac{2}{3} = \frac{5 \times 2}{7 \times 3} = \frac{10}{21}$$

So the commutative property for multiplication applies even when we deal with rational numbers.

Example 11 shows us that the multiplication is commutative because whole number multiplication is commutative. For example;
$2 \times 5 = 5 \times 2$ and
$3 \times 7 = 7 \times 3$.

Note:
$\frac{2}{3} \times \frac{5}{7}$ means $\frac{2}{3}$ of $\frac{5}{7}$ while $\frac{5}{7} \times \frac{2}{3}$ means $\frac{5}{7}$ of $\frac{2}{3}$. These sound like different ratios, but we've just shown they're the same. For example, if in Example 8 your friend could buy the coat for $\frac{2}{3}$ of the regular price and he sold it to you for $\frac{5}{7}$ of the price he paid, you'd still only be paying \$10 out of each \$21. Namely:

$\frac{2}{3}$ *of \$21 = \$14*

and

$\frac{5}{7}$ *of \$14 = \$10*

However, there is a difference to your friend. When he could buy it at 5/7 of the regular price he was paying \$15 rather than \$14. In other words, while the answers are the same, it is conceptually different to take 2/3 of 5/7 than to take 5/7 of 2/3.

Did you happen to notice in Example 11 that the product was less than either of the factors? That is:

$$\frac{2}{3} = \frac{2 \times 7}{3 \times 7} = \frac{14}{21}$$

$$\frac{5}{7} = \frac{5 \times 3}{7 \times 3} = \frac{15}{21}$$

while the answer was only $\frac{10}{21}$.

This is easy to explain in terms of "of". Clearly $\frac{2}{3}$ of any number is less than the number and so also $\frac{5}{7}$ of any number less than the number.

Example 12

How much is $\frac{4}{3} \times \frac{2}{5}$?

Answer: $\frac{8}{15}$

The rule for multiplication remains the same:

$$\frac{4}{3} \times \frac{2}{5} = \frac{4 \times 2}{3 \times 5} = \frac{8}{15}$$

For the first time, we've looked at a common fraction in which the numerator exceeds the denominator. There is nothing unnatural about this. Remember that $\frac{4}{3}$ means $4 \div 3$ and it is not extraordinary to want to divide a 4 inch piece of string into 3 parts of equal length.

Recall that all $\frac{4}{3}$ means is the number we must multiply by 3 to get 4 as the produc

A more serious question would be to explain $\frac{4}{3}$ in terms of the concept of "$\frac{4}{3}$ of ___". Well, suppose you borrow money from a person and tell him "I really needed that! and to show you my gratitude, for every \$3 you gave me I'll give you back \$4 in return. Suppose the person gave you \$15. Then you'd give him back \$20 in return. Why? First divide 15 by 3. This will tell you how many times he gave you \$3. For each time he gives you \$3, you're giving him \$4. In other words, you divide 15 by 3 and to get 5 and you then multiply 5 by 4 to get 20. All in all you took $\frac{4}{3}$ of 15.

That is, he gives you:
\$3 \$3 \$3 \$3 \$3 (\$15)
and you return to him:
\$4 \$4 \$4 \$4 \$4 (\$20)

$\frac{4}{3}$ of a number exceeds the number itself as we've just seen. The same thing happened in Example 12 $\frac{4}{3}$ of $\frac{2}{5}$ was $\frac{8}{15}$ while $\frac{2}{5} = \frac{2 \times 3}{5 \times 3} = \frac{6}{15}$.

If the numerator and denominator are equal, the common fraction is equivalent to 1.

This should not be a surprise. n/n means $n \div n$, and for any non-zero number, n this quotient is 1.

Example 13

How much is $\frac{5}{5} \times \frac{2}{3}$?

Answer: $\frac{2}{3}$

$$\frac{5}{5} \times \frac{2}{3} = \frac{5 \times 2}{5 \times 3} = \frac{\cancel{5} \times 2}{\cancel{5} \times 3} = \frac{2}{3}$$

In other words, multiplying 2/3 by 5/5 gives us the same answer as if we had multiplied 2/3 by 1.

Equivalently, if we divide a number into 5 equal parts and then multiply this by 5, we arrive at the original number.

It is often helpful in dealing with fractions to write 1 as $\frac{n}{n}$. In this same vein it is often helpful to write, say, 12 as $\frac{12}{1}$.

As we shall soon see, this is very helpful when we want to divide two common fractions.

Example 14

What whole number is named by $\frac{2}{1}$?

Answer: 2

$\frac{2}{1}$ is defined to mean $2 \div 1$ and we already know that this quotient is 2. In terms of the "of" interpretation, suppose we want

$\frac{2}{1}$ of 6

We'd first divide 6 by 1 (to get 6) and we'd then multiply our answer (6) by 2 to get 12. So $\frac{2}{1}$ of 6 means the same thing as 2×6.

More generally, if n is any whole number, then: $\frac{n}{1} = n$

That is, $\frac{2}{1}$ *of 6 means* $\frac{2}{1} \times 6$ *and this is the same as* 2×6

This result is quite useful.

Example 15

How much is $\frac{2}{7}$ of 16?

Answer: $\frac{32}{7}$

16 is the same as $\frac{16}{1}$

Hence we can write:

$\frac{2}{7}$ of 16 =

$\frac{2}{7}$ of $\frac{16}{1}$ =

$\frac{2}{7} \times \frac{16}{1}$ =

$\frac{2 \times 16}{7 \times 1}$ =

$\frac{32}{7}$

Because 16 is not a multipl[e] of 7, it is easier to first multiply by 2 and then divide by 7 when we want 2/7 o[f] 16. If we wanted 2/7 of 14 it would be easier to first divide by 7 and then multiply by 2.

Note:

$\frac{32}{7}$ *means* $7\overline{)32}$ = 4 R4

-28

4

In other words, $\frac{2}{7}$ of 16 is between 4 and 5. This is consistent with the facts that:

$\frac{2}{7}$ *of 14 = 4*

and

$\frac{2}{7}$ *of 21 = 6*

Since 16 is between 14 and 21, it is plausible that $\frac{2}{7}$ of 16 is between 4 and 6.

In the next module we shall introduce mixed numbers. In the language of mixed numbers we write 2/7 of 16 as: $4\frac{4}{7}$ In other words you have 4 complete "units" and 4 of the 7 you need for another complete "unit"

In concluding this Module we should also discuss how we divide one common fraction by another. To set the stage for this concept, we need some new terminology.

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Some New Terminology

If we interchange the numerator and denominator of a common fraction, the resulting common fraction is called the reciprocal of the first common fraction.

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So, for example, the reciprocal of 2/3 is 3/2.

As the next example indicates, there is an interesting relationship between a common fraction and its reciprocal.

Example 16

What is the product of $\frac{2}{3}$ and $\frac{3}{2}$?

Answer: 1

We have:

$\frac{2}{3} \times \frac{3}{2}$ =

$$\frac{2 \times 3}{3 \times 2} =$$

$$\frac{6}{6} =$$

$$1$$

Note that this result works for any number and its reciprocal. Symbolically:

$$\frac{m}{n} \times \frac{n}{m} = \frac{m \times n}{n \times m} = \frac{m \times n}{m \times n} = 1$$

****

Multiplicative Inverse Property:

For any non-zero common fraction, the product of it and its reciprocal is always 1. We refer to the reciprocal as the multiplicative inverse *of the given common fraction.*

More generally, if n is any non-zero rational number, there exists a number m such that $n \times m = 1$.

****

Recall that for any non-zero whole number, n:

$$\frac{n}{n} = 1$$

Whenever numerator and denominator are equal the common fraction is equivalent to 1.

"Reciprocal" seems to be an easier term to remember than "Multiplicative Inverse". But when we deal with rational numbers that are not expressed as common fractions, there are no numerators and denominators to interchange.

Example 17

What is the multiplicative inverse of 4?

Answer: $\frac{1}{4}$

That is, we want the number which when multiplied by 4 equals 1. For any common fraction the required number is the reciprocal of the given number.

So recall that 4 is equivalent to $\frac{4}{1}$ and the reciprocal of $\frac{4}{1}$ is $\frac{1}{4}$. As a check we have:

$$\frac{1}{4} \times 4 =$$

$$\frac{1}{4} \times \frac{4}{1} =$$

$$\frac{1 \times 4}{4 \times 1} =$$

$$\frac{4}{4} =$$

$$1$$

The final link we need is that multiplication of common fractions has the associative property. This is illustrated in our next example.

Example 18

Write each of the following as a common fraction:

(a) $(\frac{2}{3} \times \frac{5}{7}) \times \frac{4}{9}$

(b) $\frac{2}{3} \times (\frac{5}{7} \times \frac{4}{9})$

Answers: (a) $\frac{40}{189}$ (b) $\frac{40}{189}$

(a) Everything within the parentheses is viewed as one number and hence done first. So we have:

$$(\frac{2}{3} \times \frac{5}{7}) \times \frac{4}{9} =$$

$$\frac{10}{21} \times \frac{4}{9} =$$

$$\frac{40}{189}$$

$\frac{2}{3} \times \frac{5}{7} = \frac{2 \times 5}{3 \times 7}$

$\frac{10}{21} \times \frac{4}{9} = \frac{10 \times 4}{21 \times 9}$

(b) This time we get:

$$\frac{2}{3} \times (\frac{5}{7} \times \frac{4}{9}) =$$

$$\frac{2}{3} \times \frac{20}{63} =$$

$$\frac{40}{189}$$

$\frac{5}{7} \times \frac{4}{9} = \frac{5 \times 4}{7 \times 9}$

$\frac{2}{3} \times \frac{20}{63} = \frac{2 \times 20}{3 \times 63}$

The fact that the answers to (a) and (b) were the same is no coincidence. In multiplying numerators and denominators we had the right to regroup as we did. After all, the numerators and denominators are whole numbers, and we already know that for whole numbers multiplication has the associative property.

Make sure that you notice that (a) and (b) are exactly alike, except for the placement of the parentheses.

Let's apply some of these ideas in the next example.

Example 19

What common fraction is named by

$(\frac{5}{7} \times \frac{3}{2}) \times \frac{2}{3}$?

Answer: $\frac{5}{7}$

We have that:

$(\frac{5}{7} \times \frac{3}{2}) \times \frac{2}{3} =$

$\frac{5}{7} \times (\frac{3}{2} \times \frac{2}{3}) =$ — *by the associative property*

$\frac{5}{7} \times 1 =$ — *from Example 16 (the multiplicative inverse property)*

$\frac{5}{7}$ — *Review Example 10 if you don't see the last step.*

Example 20

Use the result of Example 19 to fill in the blank:

$\_\_\_\_ \times \frac{2}{3} = \frac{5}{7}$

Answer: $\frac{15}{14}$

Check:

$\frac{15}{14} \times \frac{2}{3} =$

$\frac{15 \times 2}{14 \times 3} =$

$\frac{3 \times 5 \times 2}{2 \times 7 \times 3} =$

$\frac{\cancel{3} \times 5 \times \cancel{2}}{\cancel{2} \times 7 \times \cancel{3}} =$

$\frac{5}{7}$

From Example 19 we see that we can replace the blank by $(\frac{5}{7} \times \frac{3}{2})$. This can in turn be simplified by writing

$\frac{5}{7} \times \frac{3}{2} =$

$\frac{5 \times 3}{7 \times 2} =$

$\frac{15}{14}$

Now we come to the key point. Rather than write

$\_\_\_\_ \times \frac{2}{3} = \frac{5}{7}$

we write:

$\frac{5}{7} \div \frac{2}{3} = \_\_\_\_$

This is in keeping with the agreement we made for whole numbers. Namely $\_\_ \times 2 = 6$ meant the same as $6 \div 2 = \_\_$
More symbolically:
first X ____ = second
means
second ÷ first = ____

In this form, Example 20 asked us to find

$\frac{5}{7} \div \frac{2}{3}$ and the answer was (before we simplified it)

$\frac{5}{7} X \frac{3}{2}$

If we compare the question and the answer, we notice that the first common fraction was left as is, the division sign was replaced by a multiplication sign, and the second common fraction was replaced by its reciprocal. This is true in general.

Rule For Dividing Common Fractions

To write $\frac{a}{b} \div \frac{c}{d}$ as a common fraction, we multiply $\frac{a}{b}$ by the reciprocal of $\frac{c}{d}$. In symbols:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} X \frac{d}{c}$$

That is, if we start with $\frac{c}{d}$ and multiply it by $\frac{d}{c}$ we get 1, and if we multiply 1 by $\frac{a}{b}$ we get $\frac{a}{b}$. So all in a we have:

$\frac{a}{b} = \frac{a}{b} X (\frac{d}{c} X \frac{c}{d})$, *or:*

$\frac{a}{b} = (\frac{a}{b} X \frac{d}{c}) X \frac{c}{d}$

Let's do a few examples, using the definition for dividing common fractions.

Example 21

What is the quotient when $\frac{3}{4}$ is divided by $\frac{2}{3}$?

Answer: $\frac{9}{8}$

Since we're dividing by $\frac{2}{3}$, the division problem has the form:

$$\frac{3}{4} \div \frac{2}{3}$$

Don't cancel! We're dividi not multiplying.

By the "invert and multiply" rule, this becomes:

To invert a common fraction means to take its reciproca

$$\frac{3}{4} X \frac{3}{2}$$

or

$$\frac{3 X 3}{2 X 4}$$

Another way to help visualize the answer is to think in terms of multiplication. We're being asked what we must multiply $\frac{2}{3}$ by to get $\frac{3}{4}$ as the quotient. We first multiply $\frac{2}{3}$ by $\frac{3}{2}$ to get 1; and we then multiply 1 by $\frac{3}{4}$ to get $\frac{3}{4}$. So altogether, we multiplied $\frac{2}{3}$ by $\frac{3}{2} \times \frac{3}{4}$.

That is: $\_\_\_ \times \frac{2}{3} = \frac{3}{4}$

Check: $\frac{9}{8} \times \frac{2}{3} = \frac{\cancel{3} \times 3 \times \cancel{2}}{\cancel{2} \times 2 \times 2 \times \cancel{3}} = \frac{3}{4}$

In terms of a ratio, $\frac{3}{4}$ *is* $\frac{9}{8}$ *of* $\frac{2}{3}$

<u>Note:</u>

In dealing with whole numbers, we think of division as making numbers smaller. Yet in this example, both $\frac{2}{3}$ *and* $\frac{3}{4}$ *were less than 1 while the quotient was greater than 1. The reason for this is that division still tells us the relative size of the two numbers. The fact that the answer exceeds 1 simply tells us that* $\frac{3}{4}$ *is greater than* $\frac{2}{3}$.

$$\begin{array}{r} 1\ R1 \\ 8\overline{)9} \\ -\ 8 \\ \hline 1 \end{array}$$

$\frac{3}{4} = \frac{9}{12}$ *and* $\frac{2}{3} = \frac{8}{12}$

There are a few things to keep track of when we divide. Most importantly, order makes a difference in division.

<u>Example 22</u>

What is the quotient when $\frac{2}{3}$ is divided by $\frac{3}{4}$?

Answer: $\frac{8}{9}$

This time we're dividing by $\frac{3}{4}$, so the division problem becomes:

$$\frac{2}{3} \div \frac{3}{4} =$$

$$\frac{2}{3} \times \frac{4}{3} =$$

$$\frac{2 \times 4}{3 \times 3} =$$

$$\frac{8}{9}$$

The answers to Examples 21 and 22 are reciprocals. In other words, if we change the order of the numbers in a division problem, one quotient is the reciprocal of the other.

While $\frac{3}{4}$ *is* $\frac{9}{8}$ *of* $\frac{2}{3}$, $\frac{2}{3}$ *is only* $\frac{8}{9}$ *of* $\frac{3}{4}$.

Psychological Problem

When we multiply, say, $\frac{2}{3}$ by $\frac{5}{7}$ we're taking $\frac{2}{3}$ of $\frac{5}{7}$. In taking $\frac{2}{3}$ of $\frac{5}{7}$ we ***divide*** *$\frac{5}{7}$ by 3 and then multiply this quotient by 2.*

We're only dividing by the 3 in $\frac{2}{3}$. We're ***not*** *saying that dividing by $\frac{2}{3}$ is the same as multiplying by $\frac{2}{3}$*

Because we are dividing by 3, it is tempting to think that we are doing a division problem. This makes it easy to confuse multiplication of common fractions with division of common fractions.

So try not to think in terms of "divding" when you divide fractions. Rather try to visualize, say,

$$\frac{2}{3} \div \frac{5}{7} = \_\_\_\_$$

as

$$\frac{5}{7} \times \_\_\_ = \frac{2}{3}$$

Perhaps it is more suggestiv to view it in the form:
$\frac{2}{3} \div \frac{5}{7} = \_\_$ *means*
$\frac{2}{3} = \frac{5}{7} \times \_\_\_$

In this way the relationship between division and multiplication of common fractions remains the same as their relationship when we deal with whole numbers.

Because multiplication seems more "natural" than division of common fractions, we often think of division problems in terms of multiplication.

That is, rather than divide by a common fraction, we multiply by its reciprocal.

Example 23

How much is $\frac{2}{5} \div 3$?

Answer: $\frac{2}{15}$

Method 1

$$\frac{2}{5} \div 3 =$$
$$\frac{2}{5} \div \frac{3}{1} =$$
$$\frac{2}{5} \times \frac{1}{3} =$$
$$\frac{2 \times 1}{5 \times 3} =$$
$$\frac{2}{15}$$

If you don't remember that $\frac{3}{1} = 3$, review Example 14.

Method 2

We already have discussed how dividing by 3 is the same as taking $\frac{1}{3}$ of a number. Hence we may read this problem as asking for $\frac{1}{3}$ of $\frac{2}{5}$ and this is $\frac{1 \times 2}{3 \times 5}$ or $\frac{2}{15}$

Example 24

How much is $12 \div \frac{1}{3}$?

Answer: 36 (Not 4)

Since $12 = \frac{12}{1}$, we have:

$$12 \div \frac{1}{3} =$$
$$\frac{12}{1} \div \frac{1}{3} =$$
$$\frac{12}{1} \times \frac{3}{1} =$$
$$\frac{12 \times 3}{1 \times 1} =$$
$$\frac{36}{1} =$$
$$36$$

$12 \times \frac{1}{3} = 4$

But $12 \div \frac{1}{3} =$ ____ means: $\frac{1}{3}$ of ____ $= 12$ and $\frac{1}{3}$ of 36 is 12.

If we again think of common fractions as adjectives modifying nouns, the interpretation of division may become more apparent.

Example 25

How much is $\frac{1}{10} \div \frac{1}{100}$?

Answer: 10

$\left(\frac{1}{10} \times \frac{1}{100} = \frac{1}{1,000}\right)$

We have:

$$\frac{1}{10} \div \frac{1}{100} =$$
$$\frac{1}{10} \times \frac{100}{1} =$$
$$\frac{100}{10} =$$
$$10$$

That is, $\frac{1}{10}$ is 10 times as great as is $\frac{1}{100}$.

Note:

Suppose we think of $\frac{1}{10}$ and $\frac{1}{100}$ as modifying dollars. We know that $\frac{1}{10}$ of a dollar is 10 cents and that $\frac{1}{100}$ of a dollar is 1 cent. Therefore:

$$\frac{1}{10}\text{(dollars)} \div \frac{1}{100}\text{(dollars)} =$$

$$10 \text{ cents} \div 1 \text{ cent}$$

and it is clear that 10 cents is ten times as much money as 1 cent.

In common fraction notation, we have:

$$\frac{10 \text{ cents}}{1 \text{ cent}} = 10$$

Thus we may cancel common denominations in the same way as we cancel common factors

The fact that we can cancel common denominations as well as we can cancel common factors gives us yet another way to emphasize whole-number division when we divide common fractions.

Example 26

Find the value of $\frac{2}{5} \div \frac{3}{7}$ by converting both common fractions to common denominators.

Answer: $\frac{14}{15}$

$$\frac{2}{5} = \frac{2 \times 7}{5 \times 7} = \frac{14}{35} = 14 \text{ thirty-fifths}$$

$$\frac{3}{7} = \frac{3 \times 5}{7 \times 5} = \frac{15}{35} = 15 \text{ thirty-fifths}$$

Therefore:

$$\frac{2}{5} \div \frac{3}{7} = 14 \text{ thirty-fifths} \div 15 \text{ thirty-fifths}$$

$$= \frac{14 \text{ thirty-fifths}}{15 \text{ thirty-fifths}}$$

$$= \frac{14 \cancel{\text{thirty-fifths}}}{15 \cancel{\text{thirty-fifths}}}$$

This is the same answer we'd get by:

$$\frac{2}{5} \div \frac{3}{7} =$$

$$\frac{2}{5} \times \frac{7}{3} =$$

$$\frac{2 \times 7}{5 \times 3} =$$

$$\frac{14}{15}$$

IMPORTANT NOTE

Just as in the study of whole numbers, we need common denominations when we add or subtract--but not when we multiply or divide.

The point is that we can do Example 26 without resorting to common denominations (in fact it's easier without common denominations), but by using common denominations the division process may become easier to understand.

In closing this module, we should at least do one real-life problem that uses division by common fractions.

Other problems appear in the self-tests.

Example 27

You can buy $\frac{2}{5}$ pounds of apples for 32¢. At this rate how much would a pound of apples cost?

Answer: 80¢

This is a rate problem. That is, our answer will have the label "cents per pound". As we did when we dealt with whole numbers, we replace "per" by "÷" to get:

32 cents per $\frac{2}{5}$ pounds =

32 cents ÷ $\frac{2}{5}$ pounds =

$(32 \div \frac{2}{5})$ cents per pound =

$(\frac{32}{1} \times \frac{5}{2})$ cents per pound =

$\frac{160}{2}$ cents per pound =

80 cents per pound.

$\frac{160}{2}$ means 160 ÷ 2, or $\frac{1}{2}$ of 160

Alternative Approach

If 2 fifths of a pound costs 32¢, then 1 fifth costs $\frac{1}{2}$ of 32¢ or 16¢. If $\frac{1}{5}$ of a pound costs 16¢, a whole pound will cost 16¢ five times or 80¢.

1 fifth is half of 2 fifths

5 fifths of a pound equals 1 pound. That is:

$$\frac{5}{5} = 1$$

In solving Example 27 either approach is fine. The alternative approach emphasizes the logic, but the division method allows you to solve the problem by using a rather simple "recipe".

The optimal approach is to understand the logic but use the recipe to do the computation more quickly.

This completes our study of common fractions. However, it does not complete our study of rational numbers. The point is that rational numbers are a concept, but common fractions are but one of several languages that are used to study rational numbers.

Among the other languages that are used are: mixed numbers, percents, and decimal fractions. Mixed numbers and percents are the subject of the next module and decimal fractions are treated in Module 7.

At this point it may be helpful to summarize the rules of arithmetic for common fractions.

(1) To add (or subtract) two common fractions rewrite the problem (if necessary) in an equivalent form in which the denominators of both fractions are the same. Then add (or subtract) the numerators and keep the common denominator.

$$\frac{3}{7} + \frac{2}{7} = \frac{5}{7}$$

$$\frac{3}{7} - \frac{2}{7} = \frac{1}{7}$$

$$\frac{3}{5} + \frac{2}{7} = \frac{21}{35} + \frac{10}{35} = \frac{31}{35}$$

(2) To multiply two common fractions, simply multiply their numerators to get the numerator of the product and multiply their denominators to get the denominator of the product. WE DO NOT NEED COMMON DENOMINATORS IN ORDER TO MULTIPLY.

$$\frac{3}{5} X \frac{2}{7} = \frac{3 X 2}{5 X 7} = \frac{6}{35}$$

(3) To divide by a fraction, you need only multiply by its reciprocal. THAT IS, EVERY DIVISION PROBLEM MAY BE REPLACED BY AN EQUIVALENT MULTIPLICATION PROBLEM.

$$\frac{3}{5} \div \frac{2}{7} =$$

$$\frac{3}{5} X \frac{7}{2} =$$

$$\frac{3 X 7}{5 X 2} =$$

$$\frac{21}{10}$$

Appendix For Module 5

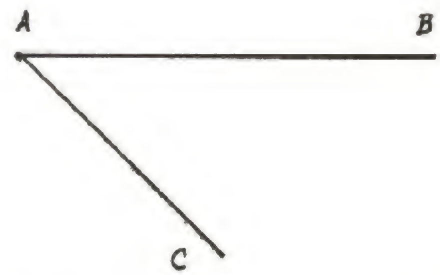
Viewing Rational Numbers in Terms of Lengths

The ancient Greeks viewed numbers as lengths. This was quite natural because of their fondness for geometry. They could use geometry to divide any length into any given number of pieces of equal length.

As a particular example suppose we want to divide a given length into 5 equal parts. Let's label the end points of the line A and B.

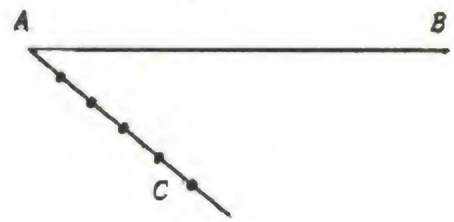
Step 1

Through either end-point (we'll choose A but it doesn't matter) draw any other line.



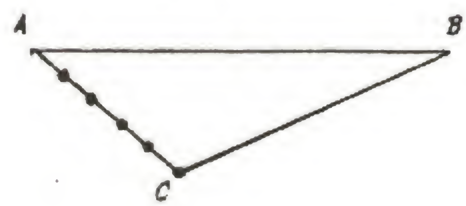
Step 2

Using any measurement you want, mark off this length 5 times on the line you drew, calling the last point C.



Step 3

Connect C to B



Step 4

Through each of the other 4 points you marked off, draw lines parallel to the line that poined C to B. Becaue AC is divided into 5 equal parts, so also is AB.

This procedure allows us to divide any length into any number of equal parts.